

Intermittent scaling of the probability density function tail of the Kardar-Parisi-Zhang equation

J. ANDERSON¹ and J. JOHANSSON²

¹ *Department of Applied Physics, Nuclear Engineering, Chalmers University of Technology, Göteborg, Sweden*

² *Lund University, Solid State Physics, Box 118, S-22100, Lund, Sweden*

PACS 05.40.-a – First pacs description
PACS 05.70.Ln – Second pacs description
PACS 68.35.Fx – Third pacs description

Abstract. – This letter provides the first analytical estimation of the probability density function (PDF) tail of the interface width in the Kardar-Parisi-Zhang equation in the regime of strongly non-linear growth. We find that the PDF tail is proportional to $\exp\{-cw_2^{3/2}\}$. In addition, the effect of spatial dimensions on the PDF tail scaling is discussed. The PDF tail is computed by the instanton method within the Martin-Rose-Siggia framework using a careful treatment of the non-linear term. This gives a novel approach to understand the rightmost PDF tail of the interface width distribution and the analysis suggests that there is no limit in the upper critical dimension.

Introduction. – In nature there are many important phenomena that are driven far from equilibrium by instabilities or by external forces. Examples are diverse from forest fires to interstellar turbulence, which is constantly stirred by supernova explosions. A proper description and understanding of the multiscale interactions that are responsible for the inevitably complex dynamics in these nonequilibrium systems remains a significant challenge in classical physics.

The out of equilibrium interfacial growth is another example that has attracted much attention during recent years. A description of these growth processes that have been widely recognized is a Langevin like equation by Kardar-Parisi-Zhang (KPZ) [1]. The KPZ equation is one of the simplest non-linear generalizations of the diffusion equation and is thus connected to many other areas of non-equilibrium dynamics such as the Burgers turbulence [2]-[3], driven diffusion and dissipative transport [4] as well as flame front propagation [5].

The KPZ equation has been studied extensively, however there are some remaining controversial issues, in particular estimates of the upper critical dimension are in the range $d_c = 2.8 - \infty$ [6]-[13]. The reason for searching for a particular value of the upper critical dimension (i.e. below d_c there is no phase transition) is the hope of systematic expansions in $d_c - d$ in analogy with equilibrium critical phenomena.

The purpose of the present work is to provide a sta-

tistical theory of interfacial growth in the strongly non-linear growth phase and thereby shed light on the elusive finite upper critical dimension in the KPZ equation. We compute the tail of the probability density function (PDF) of the interface width using the instanton method in the Martin-Siggia-Rose framework [21]. The instanton method is a non-perturbative way of computing the PDF tails [22]-[27]. The PDF tails can be viewed as a transition amplitude from a quiescent state (where no growth occurs) to a final state determined by a coherent structure, from which the PDF tails are computed through a path-integral. Similarly to the Burgers equation the KPZ equation supports a coherent structure solution that can be used in the calculation as the path with highest probability and subsequently the path-integral can be solved using the saddle-point method. The instanton predictions of the PDF tails have been shown to agree very well with numerical simulations in a model of self-organization of sheared flows [25] and heat flux [26].

The PDF has been computed in a similar manner previously in Refs. [14]-[16], however all these results relies on the assumption of a weakly non-linear system. In the present setting we focus on the effects in the intermittent or strongly non-linear regime.

We find that the PDF exhibit heavier tails than a Gaussian distribution whereas the tails are subdominant to the exponential distribution. Under the assumption of isotropic growth in d -dimensions we find a smooth vari-

ation as the dimensionality d is increased. This suggests that there exists no upper critical dimension and this is in accordance with the numerical work in Ref. [6].

The letter is organized as follows. First, the Kardar-Parisi-Zhang model is introduced and the PDF tail is computed using the instanton method. Then the result of the instanton PDF tails are corroborated in the one dimensional case through a derivation using the Fokker-Planck method. Finally the generalization to higher spatial dimensions are introduced and the letter is concluded by results and discussion.

The PDF tail of the KPZ equation. – Lateral interface growth can be described by the Kardar-Parisi-Zhang (KPZ) model [1], that is the time evolution of the height h is proportional to the square of the height gradient modified by diffusion and a stochastic forcing,

$$\frac{\partial h}{\partial t} = \lambda(\nabla h)^2 + \nu \nabla^2 h + f. \quad (1)$$

Here f is a white noise forcing with a short correlation time modeled by the delta function as

$$\begin{aligned} \langle f(x, t) f(x', t') \rangle &= \frac{1}{\epsilon \sqrt{\pi}} \delta(t - t') \exp\{-(x - x')^2 / \epsilon^2\} \\ &= \delta(t - t') \kappa_\epsilon(x - x'), \end{aligned} \quad (2)$$

and $\langle f \rangle = 0$. The angular brackets denote the average over the statistics of the forcing f . The 1+1 dimensional KPZ equation is equivalent to the noisy Burgers equation by the relations

$$u = -\nabla h, \quad (3)$$

$$h = -\int u dx. \quad (4)$$

Using the relations in Eq. (3)-(4), we now find the noisy Burgers equation to be

$$\frac{\partial u}{\partial t} + 2\lambda u \nabla u - \nu \nabla^2 u = -\nabla f. \quad (5)$$

The Burgers equation is known to support the ramp and shock-like stationary coherent solutions [2]. We will adopt the ramp solution of the form $u \propto x$, using the relations between Burgers and KPZ we can find the solution for h of the form $h = ax^2 + b$ where the time evolution of h and the non-linear term both will be proportional to x^2 . The constants a and b can be determined by inputting the trial function into KPZ yielding $a = \frac{1}{4\lambda}$ and $b = \frac{\nu}{4\lambda}$. Using the parabola solution as a coherent structure for the instanton method we will now compute the PDF tails.

We compute the PDF tails of the mean square height fluctuations (the interface width) where we give special attention to the effect of dimensional scaling by using the instanton method. Here the mean square height fluctuations are defined as

$$w_2 = \frac{1}{A_L} \sum_r (h(r, t) - \hat{h})^2 \quad (6)$$

where A_L is the area of the substrate with characteristic linear dimension L , and $\hat{h} = \sum_r h(r, t) / A_L$ is the average height of the surface. The PDF tails of w_2 are expressed in terms of a path-integral using the Gaussian statistics of the forcing. The optimum path is then associated with the creation of a short lived coherent structure (among all possible paths) and the action is evaluated using the saddle-point method on the effective action. The saddle-point solution of the dynamical variable $h(x, t)$ of the form $h(x, t) = F(t)\phi(x)$ is called an instanton if $F(t) = 0$ at $t = -\infty$ and $F(t) \neq 0$ at $t = 0$. Note that, the function $\phi(x)$ here represents the spatial form of the coherent structure. We will first consider the $d = 1$ case and then generalize the found PDF to arbitrary dimension d . The probability density function of w_2 can be defined as

$$P(w_2) = \langle \delta(M(h) - w_2) \rangle = \int d\xi e^{i\xi w_2} I_\xi, \quad (7)$$

where

$$I_\xi = \langle \exp(-i\xi M(h)) \rangle. \quad (8)$$

Here $M(h)$ is the general expression for the m -th moment (h^m), however we will restrict this study to the second moment. Following Ref. [22] the integrand can then be rewritten in the form of a path-integral as

$$I_\xi = \int \mathcal{D}h \mathcal{D}\bar{h} e^{-S_\xi}, \quad (9)$$

where effective action S_ξ of the KPZ equation is expressed as

$$\begin{aligned} S_\xi &= -i \int dx d\bar{t} \bar{h} \left(\frac{\partial h}{\partial t} - \lambda(\nabla h)^2 - \nu \nabla^2 h \right) \\ &+ \int dx dx' d\bar{t} \bar{h}(x, t) \kappa_\epsilon(x - x') \bar{h}(x', t) \\ &+ i\xi \int dx d\bar{t} h^2 \delta(t). \end{aligned} \quad (10)$$

Utilizing the instanton function $h(x, t) = F(t)\phi(x)$ the action S_ξ can now be recast into,

$$\begin{aligned} S_\xi &= -i \int dt \mu \left(c_1 \dot{F} - c_0 \lambda F^2 - c_2 \nu F \right) \\ &+ c_4 \int dt \mu^2 \\ &+ i\xi \int dt c_3 F^2 \delta(t). \end{aligned} \quad (11)$$

The conjugate variable is denoted $\bar{h} = \mu(t)\bar{\phi}$ and we have used the following definitions of the constants,

$$c_0 = \int dx \bar{\phi}(x) (\nabla \phi)^2(x) \quad (12)$$

$$c_1 = \int dx \bar{\phi}(x) \phi(x), \quad (13)$$

$$c_2 = \int dx \bar{\phi}(x) \nabla^2 \phi(x), \quad (14)$$

$$c_3 = \int dx \phi^2(x), \quad (15)$$

$$c_4 \approx \int dx dy \bar{\phi}(x) \bar{\phi}(y), \quad (16)$$

Note that the constant c_4 is evaluated for small values of ϵ in Eq. (2) and in the higher dimensional case c_2 will change with the dimension d and $c_2 \neq c_0$. We compute the first variational derivatives to minimize S_ξ with respect to F and μ in order to find the path with highest probability identified by the instanton or the extremum of the action as,

$$\frac{\delta S_\xi}{\delta \mu} = -i \left(c_1 \dot{F} - c_0 \lambda F^2 - \nu c_2 F \right) + 2c_4 \mu = 0, \quad (17)$$

$$\frac{\delta S_\xi}{\delta F} = -i \left(-c_1 \dot{\mu} - 2c_0 \lambda F \mu - \nu c_2 \mu \right) + 2i \xi c_3 F \delta(t) = 0. \quad (18)$$

Eqs. (17) - (18) constitutes a dynamical system for the instanton time function F and its conjugate μ . We proceed by solving these equations for $t < 0$ and matching the solution at $t = 0$. Note that the instanton solution F rapidly grows at $t = 0$ with increasing ξ while it vanishes as $t \rightarrow -\infty$. We start by computing an additional relation for the time evolution of the conjugate variable μ expressed in the real variable F by taking the time derivative on Eq. (17),

$$c_1 \ddot{F} - 2c_0 \lambda \dot{F} F - \nu c_2 \dot{F} = -2i c_4 \dot{\mu}. \quad (19)$$

We now substitute μ and $\dot{\mu}$ in Eq. (19) by using Eqs. (17) and (18) yielding a second order non-linear differential equation for F ,

$$c_1^2 \ddot{F} = \nu^2 c_2^2 + 3\nu c_0 c_2 \lambda F^2 + 2c_0^2 \lambda^2 F^3. \quad (20)$$

By setting $v = \dot{F}$ we find that we can rewrite the time derivative as $v(dv/dF)$ and we can now perform the first integration,

$$\begin{aligned} c_1^2 v^2 &= \nu^2 c_2^2 F^2 + 2\nu c_0 c_2 \lambda F^3 + c_0^2 \lambda^2 F^4 \\ &= F^2 (\nu c_2 + c_0 \lambda F)^2. \end{aligned} \quad (21)$$

Below we will utilize the relation between \dot{F} and F that can be expressed as,

$$c_1 \dot{F} = \pm F (\nu c_2 + c_0 \lambda F). \quad (22)$$

We can now easily determine the instanton time dependence from the separable differential Eq. (22),

$$F(t) = \frac{\nu c_2}{H - c_0 \lambda}, \quad (23)$$

$$H(t) = \frac{\nu c_2 + c_0 \lambda F(0)}{F(0)}. \quad (24)$$

Note that the equation for $F(t)$ gives $F(0)$ at $t = 0$. We now have to determine a value of F at $t = 0$ as a function

of ξ . Thus we integrate Eq. (18) over $(-\epsilon, \epsilon)$ and use Eqs. (17) and (22) with the negative sign to obtain,

$$\mu(0) \approx 2i \frac{c_3 c_4}{c_0 c_1 \lambda} \xi. \quad (25)$$

The path-integral will now be computed using the saddle-point method in the limit of $\xi \rightarrow \infty$. First we have to evaluate the ξ -dependence of the action S_ξ . In the limit of $\xi \rightarrow \infty$, S_ξ becomes

$$\begin{aligned} S_\xi &= -i \int dt \mu \left(c_1 \dot{F} - c_0 \lambda F^2 - \nu c_2 F \right) \\ &+ c_4 \int dt \mu^2 \\ &+ i \xi \int dt c_3 F^2 \delta(t) \\ &= \frac{1}{4c_4} \int dt (c_1 \dot{F} - c_0 \lambda F^2 - \nu c_2 F)^2 + i \xi c_3 F^2(0) \\ &= -\frac{c_1}{c_4} \int_0^{F(0)} dF (\nu c_2 F + c_0 \lambda F^2) + i \xi c_3 F^2(0) \\ &= -\frac{c_1}{c_4} \left(\nu c_2 \frac{F^2(0)}{2} + c_0 \lambda \frac{F^3(0)}{3} \right) + i \xi c_3 F^2(0) \\ &\approx \frac{8}{3} i \frac{c_3^3 c_4^2}{c_0^2 c_1^2 \lambda^2} \xi^3 - 4i \frac{c_3^3 c_4^2}{c_0^2 c_1^2 \lambda^2} \xi^3 \\ &= -\frac{4}{3} i \frac{c_3^3 c_4^2}{c_0^2 c_1^2 \lambda^2} \xi^3 \end{aligned} \quad (26)$$

Now let

$$\zeta = \frac{4}{3} \frac{c_3^3 c_4^2}{c_0^2 c_1^2 \lambda^2}, \quad (27)$$

and the action becomes $S_\xi = -i \zeta \xi^3$. The tail of the PDF is then found by performing the ξ -integral in Eq. (7) by the saddle point method in the limit $w_2 \rightarrow \infty$. It is later shown that this corresponds to $\xi \rightarrow \infty$,

$$P(w_2) \sim \int d\xi e^{i \xi w_2 - S_\xi} \quad (28)$$

$$\approx e^{i \xi w_2 + i \zeta \xi^3}. \quad (29)$$

We evaluate the ξ -integral using the extreme point $f'(\xi_0) = 0$ and $\xi_0^2 = -R/(3\zeta)$ of $f(\xi) = i \xi w_2 + i \zeta \xi^3$ for the saddle-point method. This results in the PDF of w_2 as,

$$P(w_2) \sim \exp \left\{ -\frac{2}{3} \frac{w_2^{3/2}}{\sqrt{3\zeta}} \right\} \quad (30)$$

where ζ is determined by Eq. (27).

Computing the PDF tails using the Fokker-Planck equation. – In order to validate our found PDF we perform a similar analysis for $d = 1$ where we compute the PDF using the Fokker-Planck (FP) method [22], [27]. However, it seems to be very difficult to generalize the

PDFs found using the FP method to arbitrary spatial dimensions. Furthermore, the PDF is found to have the same fundamental exponential form whereas the coefficients differ. We assume that we can write the functions (h and f) in Eq. (1) as

$$h(x, t) = \phi(x)F(t), \quad (31)$$

$$f(x, t) = \phi(x)g(t), \quad (32)$$

with ϕ as before and $\langle g(t)g(t_1) \rangle = G\delta(t-t_1)$. By substituting this into Eq. (1) we find (neglecting the dissipation),

$$\dot{F} = 4\lambda a F^2 + g(t). \quad (33)$$

Furthermore we write the generating function (z) and the PDF (P) as

$$z = e^{i\xi F}, \quad (34)$$

$$\langle z \rangle = \int dF P(F) e^{i\xi F} = \tilde{P}(\xi), \quad (35)$$

$$P(F) = \int d\xi e^{i\xi F} \langle z \rangle. \quad (36)$$

Here the $\langle \cdot \rangle$ is the mean value integral over the forcing f . To compute the PDF from the FP equation we have to determine the time evolution of the generating function z using Eq. (34),

$$\frac{\partial z}{\partial t} = i\xi \dot{F} e^{i\xi F} \quad (37)$$

$$= i\xi (4\lambda a F^2 + g(t)) z. \quad (38)$$

The time evolution of P can now be found using Eq. (34) - (36) as

$$\frac{\partial P}{\partial t} = -4\lambda a \frac{\partial}{\partial F} (F^2 P) - G \frac{\partial^2}{\partial F^2} P. \quad (39)$$

In the case of stationary PDF the differential equation is separable and we find

$$P(F) = P_0 e^{-\frac{4\lambda a}{G} F^3}, \quad (40)$$

where this can be rewritten in terms of \hat{h}^2 and we find

$$P(\hat{h}^2) = P_0 e^{-\frac{4\lambda a}{G} (\hat{h}^2)^{3/2}}. \quad (41)$$

This is the same stretched exponential dependence as was found using the instanton method whereas the coefficient is different in the case of one spatial dimension.

Dimensional scaling of the PDF tail. – In this letter we will model the effects of several spatial dimensions for the PDFs found from the instanton method as being isotropic in all directions. Note that the KPZ has two terms involving the gradient operator where we assume $\nabla \phi(x_1, x_2, \dots, x_d) \approx k \phi(x_1, x_2, \dots, x_d)$ where k_d is the spatial length scale and ϕ is the spatial coherent structure in d dimensions. In the isotropic case the terms become $\nabla^2 \phi = dk_d^2 \phi$ and $(\nabla \phi)^2 = dk_d^2 \phi^2$. The generalization to

the anisotropic case is straightforward, however, it does not add significantly to the understanding of scalings of dimensionality in the PDF. The coefficients in the found PDF transforms as,

$$c_{0d} = dk_d^2 c_0, \quad (42)$$

$$c_{1d} = c_1, \quad (43)$$

$$c_{2d} = dk_d^2 c_2. \quad (44)$$

Here the coefficients c_{0d}, c_{1d}, c_{2d} represents the transformed coefficients in the d -dimensional case. We will now show some results of the predicted PDF tails combining Eq. (30) with Eqs. (42)-(44).

We will now elucidate on the effects of in particular dimensionality but also on the significance of the non-Gaussian tails by computing the dimensional dependence of the PDF tails. For the case of dimension $d = 1$ we will compare the analytical prediction in Eq. (30) with an exponential tail, $P(w_2) \sim e^{-cdw_2^2}$, which we motivate below. The expression for the PDF tails can be written as

$$P(w_2) \sim e^{-cdw_2^{3/2}}, \quad (45)$$

where c is a numerical coefficient and d is the spatial dimension. Note that the power $3/2$ in the exponential comes directly from the non-linear interaction term in the KPZ equation whereas the factor cd arises from the linear part of the equation. This suggests that KPZ supports harmonic growth in all spatial directions and the physical dimensions are weakly correlated with a smooth behaviour of the scaling function as the dimensionality changes. Furthermore, it is important to note that to have a finite upper critical dimension, the scale lengths k_d in Eqs. (42) and (44) have to change.

Much of the knowledge about the KPZ equation come from simulations on systems, typically lattice models, that belong to the KPZ universality class. The problem with such lattice models is that given a specific model, such as the restricted solid-on-solid model [28], the KPZ parameters, λ and ν are fixed and the non-linear parameter, λ , is often quite small [29], which result in PDF tails that are indistinguishable from exponentially decaying PDF tails [16] or result in stretched exponentials with exponent slightly smaller than one [30]. This is in contrast to our exponent of $3/2$, which holds in the strongly non-linear regime. Due to numerical instabilities, which seem to grow with increasing λ , it is a very challenging task to solve the KPZ equation by numerical integration. Thus, we are not aware of any numerical estimations of the solutions to the KPZ equation in the strongly non-linear regime with which we could compare our analytical results.

Results and discussions. – In finding the analytical solution for the PDF we assumed that we solved for values in the rightmost part of the tail due to taking $\xi \rightarrow \infty$ and using the saddle-point method whereas here we will show the PDF for all values of $w_2/\langle w_2 \rangle > 1$. The constant c in Eq. (45) includes an unknown forcing strength and

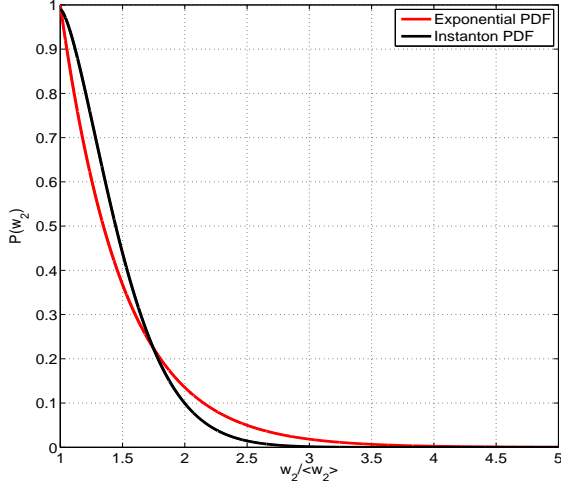


Fig. 1: (Color online). The scaling function of the width distribution $P(w_2)$ as a function of $w_2/\langle w_2 \rangle$ (linear scale) where the results are shown for exponential PDF (red) and instanton PDF (black).

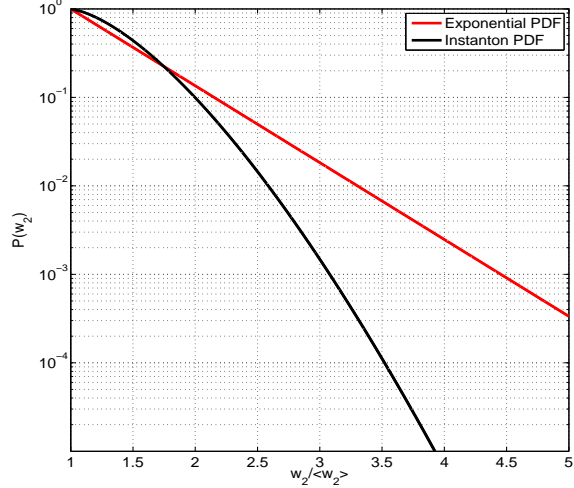


Fig. 2: (Color online). The scaling function of the width distribution $w_2 P(w_2)$ as a function of $w_2/\langle w_2 \rangle$ (log-linear scale) where the results are displayed for exponential PDF (red) and instanton PDF (black).

can be estimated from simulations, e.g. those reported in Ref. [6]. In Figure 1 we show the scaling function of the width distribution $P(w_2)$ as a function of $w_2/\langle w_2 \rangle$ where quite small differences are present. Note that the PDF is normalized in such a way that the total probability is unity and that $P(1) \approx 1$ whereas the values of $w_2/\langle w_2 \rangle$ are arbitrary. The maximum value of the PDF is determined by the strength of the forcing. Figure 2 displays again the scaling function $P(w_2)$ as a function of $w_2/\langle w_2 \rangle$ however this time in lin-log scale which highlights the significant differences for small probabilities.

To this end, in this paper we have computed the PDF tails in the KPZ model and evaluated the dependence of spatial dimensionality. The PDF tails of the form $\sim \exp\{-cw_2^{3/2}\}$ (where c is a numerical coefficient dependent on the dimension d) are relevant for intermittent events in the strongly non-linear growth regime. Moreover, the PDF tails are significantly alleviated compared to a Gaussian distribution whereas the tails are subdominant to the exponential distribution. Of particular interest are the effects of higher dimensions on the coefficient c . Here it is shown that using an isotropic growth model the coefficient c changes smoothly as d increases and thus suggests that there is no upper critical dimension d_c .

* * *

Johansson performed his part of the work at the Nanometer Structure Consortium at Lund University (nmC@LU) and wants to acknowledge the Swedish Research Council (VR) for financial support and Anderson acknowledges the Max-Planck Institute for its hospitality.

REFERENCES

- [1] M. Kardar, G. Parisi and Y. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986)
- [2] J. M. Burgers, The Nonlinear Diffusion Equation (Riedel, Boston, 1974)
- [3] D. Forster, D. R. Nelson and M. J. Stephen, Phys. Rev. A **16**, 732 (1977)
- [4] H. van Beijeren, R. Kutner and H. Spohn, Phys. Rev. Lett. **54**, 2026 (1985)
- [5] Y. Kuramoto and T. Tsuzuki, Prog. Theor. Phys. **55**, 356 (1977)
- [6] E. Marinari, A. Pagnani, G. Parisi and Z. Racz, Phys. Rev. E **65**, 026136 (2002)
- [7] T. Halpin-Healy, Phys. Rev. A **42**, 711 (1990)
- [8] J. Cook and B. Derrida, J. Phys. A **23**, 1523 (1990)
- [9] M. Schwartz and S. F. Edwards Europhys. Lett. **20**, 301 (1992)
- [10] J-P. Bouchaud and M. E. Cates, Phys. Rev. E **47**, 1455 (1993)
- [11] J. P. Doherty, M. A. Moore, J. M. Kim, and A. J. Bray, Phys. Rev. Lett. **72**, 2041 (1994)
- [12] Y. Tu, Phys. Rev. Lett. **73**, 3109 (1994)
- [13] F. Colaiori and M. A. Moore, Phys. Rev. Lett. **86**, 3946 (2001)
- [14] Z. Racz and M. Plischke, Phys. Rev. E **50**, 3530 (1994)
- [15] M. Plischke, Z. Racz and R. K. P. Zia, Phys. Rev. E **50**, 3589 (1994)
- [16] G. Foltin, K. Oerding, Z. Racz *et al*, Phys. Rev. E **50**, R639 (1994)
- [17] T. Sasamoto and H. Spohn, Proceedings StatPhys24, 19-23 July 2010, Cairns, Australia (<http://arxiv.org/abs/1010.2691>)
- [18] H. C. Fogedby, A. B. Eriksson and L. V. Mikheev, Phys. Rev. Lett. **75**, 1883 (1995)
- [19] S. M. A. Tabei, A. Bahraminasab, A. A. Masoudi, S. S.

- Mousavi and M. R. R. Tabar, Phys. Rev. E. **70**, 031101 (2004)
- [20] I. V. Kolokolov and S. E. Korshunov, Phys. Rev. B **78**, 024206 (2008)
 - [21] P. C. Martin, E. D. Siggia and H. A. Rose, Phys. Rev. E **8**, 423 (1973)
 - [22] J. Zinn-Justin Field Theory and Critical Phenomena (Clarendon, Oxford, 1989)
 - [23] V. Gurarie and A. Migdal, Phys. Rev. E **54**, 4908 (1996)
 - [24] G. Falkovich, I. Kolokolov, V. Lebedev and A. Migdal, Phys. Rev. E **54**, 4896 (1996)
 - [25] E. Kim, H.-L. Liu and J. Anderson, Phys. Plasmas **16**, 052304 (2008)
 - [26] J. Anderson and P. Xanthopoulos Phys. Plasmas **17**, 110702 (2010)
 - [27] E. Kim and J. Anderson, Phys. Plasmas **15**, 114506 (2008)
 - [28] K. Park and B. N. Kahng, Phys. Rev. E **51**, 796 (1995)
 - [29] J. Krug, P. Meakin, and T. Halpin-Healy, Phys. Rev. A **45**, 638 (1992)
 - [30] V. G. Miranda and F. D. A. Aarão Reis, Phys. Rev. E **77**, 031134 (2008)